Analysis on the origin of directed current from a class of microscopic chaotic fluctuations

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We show that the Perron-Frobenius equation of microscopic chaos based on double symmetric maps leads to an inhomogeneous Smoluchowski equation with a source term. Our perturbative analysis reveals that the source term gives rise to a directed current for a strongly damped particle in a spatially periodic potential. In addition, our result proves that in the zeroth-order limit, the position distribution of the particle obeys the Smoluchowski equation even though the fluctuating force is deterministic.

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I. INTRODUCTION

In accordance with the second law of thermodynamics, usable work cannot be extracted from equilibrium fluctuations. This is not the case, however, for nonequilibrium fluctuations, where rectification can turn the unbiased randomness into directed motion for useful work. Recently, such Maxwell's Demon mechanisms, also known as the ratchet effects, are of great theoretical interest, because their understanding will contribute to the design of novel artificial mesoscale devices [1], as well as the explanation of unidirectional transport in molecular motors [2–4].

Current research in these Brownian ratchet systems has led to various proposals [5], which have been classified according to whether they are being subjected to a time-varying potential [3,6], or whether an external fluctuating force has been supplied [7]. The latter type of ratchet systems are also called tilting ratchets, which draw their energy from fluctuations that are either correlated in time, or are white but non-Gaussian.

Generically, the potential of tilting ratchets are periodic and spatially asymmetric. However, it is interesting that in the more restricted case of a completely symmetric and periodic potential, work can still be performed out of the nonequilibrium fluctuations [8–11]. Physically, this is possible due to broken symmetry in the fluctuating force [12], and citing Curie's principle, a current is to be expected. Nevertheless, concrete affirmation of directed motion requires an analytical derivation for the current, which has been achieved for noise that is deterministic, periodic but time asymmetric [9]; or stationary stochastic, such as the white shot noise [10].

As is generally known, there is another source of nonequilibrium noise. This is the deterministic noise from chaotic dynamical system, which has been considered in the context of its effects on multistable system [11], as well as spatially asymmetric ratchet system [13]. In these cases, the current has been attributed to the dynamical asymmetry and deterministic property of the chaotic noise. But from the perspective of statistical symmetry breaking, the basic existence of the current can be physically explained from the nonvanishing of odd higher order correlations in the chaotic fluctuation [12,14]. In this respect, directed motion is expected in a spatially symmetric ratchet due to asymmetric chaotic noise, although the precise manner in which the microscopic chaos [15] affects the macroscopic particle transport has not been fully resolved.

In this paper, we attempt to address this problem in greater detail, through establishing an analytical expression for the directed current. This is first carried out by employing and extending a model [16], which has been used to study the relationship between microscopic chaos and Gaussian diffusion process, by including a generic potential $V(x)$ as an additional force field faced by the particle. The resulting derivation, which will be described in Sec. II, yields a nonlinear map which we name the generalized kicked particle (GKP) map. Subsequently, by treating the physical problem in the strong friction regime, the GKP map is reduced to a quasistationary version, which we call the quasistationary kicked particle (QKP) map. In Sec. III, we begin our analysis on the evolution of an ensemble of trajectories from these maps by means of the Perron-Frobenius equation [17,18], which relates the density of states of the particle at consecutive time instances. This is followed by a perturbative analysis, with τ/γ (where τ is the time interval between chaotic kicks and γ is the viscous coefficient) being the perturbative parameter. Then, in Sec. IV, with the chaotic fluctuations resulting from the class of double symmetric map [17], we show that the first-order position density function of the particle satisfies an inhomogeneous Smoluchowski equation with a source term. Physically, the source term gives rise to a directed current in a spatially periodic potential, which is shown in Sec. V. In Sec. VI, the inhomogeneous Smoluchowski equation is solved specifically for the spatially symmetric cosine potential and fluctuation based on a double symmetric map—the Ulam map [19]. Finally, Sec. VII discusses and compares the results obtained from analytical derivation as well as numerical simulation.

II. THE PHYSICAL MODEL

A. Generalized kicked particle map

We formulate a nonlinear model in which a particle under the influence of a potential $V(x)$ is being constantly subjected to an impulsive force. Denoted as $(\gamma m \tau)^{1/2} F^{\mathrm{I}}(t)$, the impulsive fluctuating force is assumed to be deterministic with nonlinear dynamical origin. Here, m denotes the mass of the particle; τ is the time interval between the kicks of the

impulsive force; and the parameter γ is the viscous friction coefficient of the medium. Accordingly, the Hamiltonian for this dynamical model with respect to the reduced phase space (x,p) is as follows:

$$H=\frac{p^2}{2m}+V(x)-(\gamma m\tau)^{1/2}xF^{\mathrm{I}}(t)\sum_n \delta(t-n\tau), \quad (1)$$

where $\delta(\cdots)$ is the Dirac's delta function. From Eq. (1), the equations of motion are

$$\frac{dx}{dt}=\frac{\partial H}{\partial p}=\frac{p}{m}, \quad (2)$$

$$\frac{dp}{dt}=-\frac{\partial H}{\partial x}=-\frac{\partial V(x)}{\partial x}+(\gamma m\tau)^{1/2}F^{\mathrm{I}}(t)\sum_n \delta(t-n\tau). \quad (3)$$

Considering the dissipative drag from the medium, the particle is also being acted upon by a viscous force F_{vis} given by

$$F_{\mathrm{vis}}=-\gamma p. \quad (4)$$

As a result, Eq. (3) becomes

$$\frac{dp}{dt}=-\gamma p-\frac{\partial V(x)}{\partial x}+(\gamma m\tau)^{1/2}F^{\mathrm{I}}(t)\sum_n \delta(t-n\tau). \quad (5)$$

Thus, if the impulsive force $(\gamma m\tau)^{1/2}F^{\mathrm{I}}(t)$ is defined appropriately, Eq. (2) and Eq. (5) will constitute a nonlinear dynamical system. Notice that if $\tau\to 0$, Eq. (5) reduces to the Langevin's equation. In addition, if $F^{\mathrm{I}}(t)$ is a Gaussian random process, this equation describes Brownian motion.

We are interested in a theory based on a series of discrete snapshots of this system. The snapshot is a phase space point immediately after the impulsive kicks. More specifically, the trajectories from the system [Eqs. (2) and (5)] will be recorded only at time instant $t=n\tau^+$, where $n\tau^+=n\tau+0^+$ (and $n\tau^-=n\tau-0^+$). This renders the continuous time dynamical system discrete; the snapshot at time $t=n\tau^+$ is expressed as (x_n,p_n). Similarly, we shall write $F^{\mathrm{I}}(n\tau^+)\approx F^{\mathrm{I}}(n\tau)=F^{\mathrm{I}}_n$.

With this definition, we proceed to solve Eqs. (2) and (5) to obtain (refer to Appendix A for details)

$$p_{n+1}=e^{-\gamma\tau}p_n-e^{-\gamma\tau}\int_{0^+}^{\tau^-}e^{\gamma t}V'[x_n(t)]dt+(\gamma m\tau)^{1/2}F^{\mathrm{I}}_{n+1}, \quad (6)$$

$$x_{n+1}=x_n+\frac{1}{m}\int_{0^+}^{\tau^-}p_n(t)dt. \quad (7)$$

From Eq. (6), we see that $(\gamma m\tau)^{1/2}F^{\mathrm{I}}_n$ is analogous to the stochastic fluctuation in Langevin's formulation. However, in this paper, we are interested in the case where F^{I}_n is deterministic in that its time evolution is governed by a nonlinear dynamical map G:

$$F_{n+1}=G(F_n), \quad (8)$$

with F_n serving as the chaotic fluctuation whose intensity is to be adjusted by a factor $s=\sqrt{2kT}/\sigma$, with k being the Boltzmann constant, T the temperature, and σ the standard deviation of the fluctuation, so that the impulsive term is modeled as

$$F^{\mathrm{I}}_{n+1}=sF_{n+1}. \quad (9)$$

Moreover, we choose a suitable G that is ergodic, and possess the property

$$\langle F_iF_j\rangle=\sigma^2\delta_{ij}, \quad (10)$$

where δ_{ij} is the Kronecker delta function, and the $\langle\cdots\rangle$ denotes expectation with respect to $h(F)$, which is the invariant density of the dynamics of G. This model of the chaotic noise, together with Eqs. (6) and (7), shall form a purely deterministic map called the generalized kicked particle (GKP) map:

$$F_{n+1}=G(F_n), \quad (11)$$

$$p_{n+1}=e^{-\gamma\tau}p_n-e^{-\gamma\tau}\int_{0^+}^{\tau^-}e^{\gamma t}V'[x_n(t)]dt+(\gamma m\tau)^{1/2}sF_{n+1}, \quad (12)$$

$$x_{n+1}=x_n+\frac{1}{m}\int_{0^+}^{\tau^-}p_n(t)dt. \quad (13)$$

B. Quasistationary kicked particle map

In the strong friction regime, the relaxation time γ^{-1} is short, which implies that the ensemble of kicked particles settles down rapidly to a stationary distribution. Accordingly, the spatial position x of the particle possesses a variation of order $(kT/m)^{1/2}\gamma^{-1}$. If the force field $V'(x)$ does not change appreciably over such a spatial scale, $V'[x_n(t)]$ can be viewed as a constant from 0^+ to τ^- when $\tau>\gamma^{-1}$, leading to a simplification of the GKP map [20]. The resulting quasistationary version of the GKP map is termed the quasistationary kicked particle (QKP) map, which is given as follows:

$$F_{n+1}=G(F_n), \quad (14)$$

$$p_{n+1}=e^{-\gamma\tau}p_n-\frac{V'(x_n)}{\gamma}(1-e^{-\gamma\tau})+(\gamma\tau)^{1/2}sF_{n+1}, \quad (15)$$

$$x_{n+1}=x_n+\frac{1}{\gamma}(1-e^{-\gamma\tau})p_n-\frac{V'(x_n)}{\gamma}\left(\tau+\frac{1}{\gamma}(e^{-\gamma\tau}-1)\right), \quad (16)$$

where, without loss of generality, we have assumed $m=1$ in the above as well as in subsequent derivation.

III. THE PERRON-FROBENIUS APPROACH

For the QKP map, let us impose the restriction $\gamma\tau\gg 1$, so that the iterated solution of the particle's position becomes

$$x_{n+1}=\left(\frac{\tau}{\gamma}\right)^{1/2} s\sum_{i=0}^{n-1} F_{i+1}-\left(\frac{\tau}{\gamma}\right)\sum_{i=0}^{n-1} V'(x_i). \tag{17}$$

This suggests that the position variable can be expressed independently of the momentum variable. By separating out the momentum variable, we arrive at a simplified map relating the chaotic fluctuation F_n and the position variable x_n,

$$F_{n+1}=G(F_n), \tag{18}$$

$$x_{n+1}=x_n+\left(\frac{\tau}{\gamma}\right)^{1/2} sF_n-\left(\frac{\tau}{\gamma}\right)V'(x_n). \tag{19}$$

For the sake of notational convenience, we shall also express this map succinctly as $(\overline{F},\overline{x})=f(F,x)$, where the overbar represents the states F and x at the next instant.

Instead of examining a single trajectory from this map, let us study an ensemble of trajectories, for example, those in the range $(\overline{F},\overline{x})$ to $(\overline{F}+d\overline{F},\overline{x}+d\overline{x})$ at time $(n+1)\tau$ in phase space. These trajectories are found to map deterministically from the set of ranges $(F^{(i)},x^{(i)})$ to $(F^{(i)}+dF^{(i)},x^{(i)}+dx^{(i)})$ at time $n\tau$ through the relation $(\overline{F},\overline{x})=f(F,x)$. This has the implication that the density of trajectories in the interval $(\overline{F},\overline{x})$ to $(\overline{F}+d\overline{F},\overline{x}+d\overline{x})$ at time $(n+1)\tau$ is the sum of the densities in the intervals $(F^{(i)},x^{(i)})$ to $(F^{(i)}+dF^{(i)},x^{(i)}+dx^{(i)})$ at time $n\tau$, which can be expressed mathematically as

$$\rho_{n+1}(\overline{F},\overline{x})=\sum_{(F,x)\in f^{-1}(\overline{F},\overline{x})}\frac{1}{|\det Df|}\rho_n(F,x), \tag{20}$$

where $\rho_n(F,x)$ is a probability measure that describes the density of trajectories in (F,x) at $n\tau$.

Equation (20) is called the Perron-Frobenius equation. It describes the conservation as well as evolution of the probability density ρ_n of a deterministic dynamical system. In this section, we will show that by performing a perturbative expansion of the Perron-Frobenius equation, an equation of the Fokker-Planck type shall arise from the nonlinear dynamics given by Eqs. (18) and (19). In the following derivation, we will let τ/γ to be small or $\tau/\gamma\to 0$, which is a reasonable assumption in the strong friction regime. With τ/γ being a small parameter, a perturbative expansion up till the order $O((\tau/\gamma)^{3/2})$ will be carried out, as this is sufficient for our purpose.

First, let us evaluate $1/|\det Df|$ of the Perron-Frobenius equation by employing Eq. (19):

$$\begin{aligned}\frac{1}{|\det Df|}&=\frac{1}{|G'(F)|}\left|1-\left(\frac{\tau}{\gamma}\right)V''(x)\right|^{-1}\\&=\frac{1}{|G'(F)|}\left[1+\left(\frac{\tau}{\gamma}\right)V''(\overline{x})-\left(\frac{\tau}{\gamma}\right)^{3/2}sFV'''(\overline{x})\right.\\&\quad\left.+O\left(\left(\frac{\tau}{\gamma}\right)^2\right)\right].\end{aligned} \tag{21}$$

Notice that the potential $V(x)$ is assumed to be analytic. Moreover, it is expressed as a function of x at the next instant, so that consecutive time instants of x are not involved during the evaluation of the Perron-Frobenius equation later. Reexpressing Eq. (19) in the same vein, we have

$$x=\overline{x}-\left(\frac{\tau}{\gamma}\right)^{1/2}sF+\left(\frac{\tau}{\gamma}\right)V'(\overline{x})-\left(\frac{\tau}{\gamma}\right)^{3/2}sFV''(\overline{x})+O\left(\left(\frac{\tau}{\gamma}\right)^2\right). \tag{22}$$

Putting these results into Eq. (20) and then Taylor expand the Perron-Frobenius equation in terms of τ/γ, we obtain a perturbative version

$$\begin{aligned}\rho_{n+1}(\overline{F},x)&=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\left[1+\left(\frac{\tau}{\gamma}\right)V''(x)\right.\\&\quad\left.-\left(\frac{\tau}{\gamma}\right)^{3/2}sFV'''(x)\right]\rho_n\left(F,x-\left(\frac{\tau}{\gamma}\right)^{1/2}sF\right.\\&\quad\left.+\left(\frac{\tau}{\gamma}\right)V'(x)-\left(\frac{\tau}{\gamma}\right)^{3/2}sFV''(x)\right)\\&=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\left\{\rho_n(F,x)-\left(\frac{\tau}{\gamma}\right)^{1/2}sF\frac{\partial\rho_n}{\partial x}\right.\\&\quad+\left(\frac{\tau}{\gamma}\right)\left[V'(x)\frac{\partial\rho_n}{\partial x}+\frac{1}{2}s^2F^2\frac{\partial^2\rho_n}{\partial x^2}+V''(x)\rho_n\right]\\&\quad-\left(\frac{\tau}{\gamma}\right)^{3/2}\left[2sFV''(x)\frac{\partial\rho_n}{\partial x}+sFV'(x)\frac{\partial^2\rho_n}{\partial x^2}\right.\\&\quad\left.\left.+\frac{1}{6}s^3F^3\frac{\partial^3\rho_n}{\partial x^3}+sFV'''(x)\rho_n\right]+O\left(\left(\frac{\tau}{\gamma}\right)^2\right)\right\}.\end{aligned} \tag{23}$$

Following Beck [17], we introduce a continuous-time smooth suspension $\rho(F,x,t)$,

$$\rho(F,x,t):=\rho_n(F,x). \tag{24}$$

Note that this suspension is deemed to be true only at stroboscopic time $t=n\tau$. Representing the left hand side of the Perron-Frobenius equation by this suspension, which is assumed to be analytic, a Taylor expansion yields

$$\begin{aligned}\rho_{n+1}(\overline{F},x)&=\rho(\overline{F},x,n\tau+\tau)\\&=\rho(\overline{F},x,t)+\tau\frac{\partial}{\partial t}\rho(\overline{F},x,t)\\&\quad+\frac{1}{2}\tau^2\frac{\partial^2}{\partial t^2}\rho(\overline{F},x,t)+\cdots.\end{aligned} \tag{25}$$

Let us consider the following perturbative ansatz:

$$\rho(F,x,t)=\rho^{(0)}(F,x,t)+\left(\frac{\tau}{\gamma}\right)^{1/2}q^{(1)}(F,x,t)+\left(\frac{\tau}{\gamma}\right)q^{(2)}(F,x,t)+\left(\frac{\tau}{\gamma}\right)^{3/2}q^{(3)}(F,x,t)+O\left(\left(\frac{\tau}{\gamma}\right)^{2}\right). \tag{26}$$

Substituting this ansatz into Eq. (25), we get

$$\rho_{n+1}(\overline{F},x)=\rho^{(0)}(\overline{F},x,t)+\tau^{1/2}\gamma^{-1/2}q^{(1)}(\overline{F},x,t)+\tau\left[\gamma^{-1}q^{(2)}(\overline{F},x,t)+\frac{\partial}{\partial t}\rho^{(0)}(\overline{F},x,t)\right]+\tau^{3/2}\left[\gamma^{-3/2}q^{(3)}(\overline{F},x,t)+\gamma^{-1/2}\frac{\partial}{\partial t}q^{(1)}(\overline{F},x,t)\right]+O(\tau^{2}). \tag{27}$$

Also, in terms of Eq. (26), the perturbative Perron-Frobenius equation (23) becomes

$$\begin{aligned}\rho_{n+1}(\overline{F},x)=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\Bigg\{&\rho^{(0)}+\left(\frac{\tau}{\gamma}\right)^{1/2}\left[q^{(1)}-sF\frac{\partial\rho^{(0)}}{\partial x}\right]+\left(\frac{\tau}{\gamma}\right)\left[q^{(2)}-sF\frac{\partial q^{(1)}}{\partial x}+V'(x)\frac{\partial\rho^{(0)}}{\partial x}+\frac{1}{2}s^{2}F^{2}\frac{\partial^{2}\rho^{(0)}}{\partial x^{2}}+V''(x)\rho^{(0)}\right]\\&+\left(\frac{\tau}{\gamma}\right)^{3/2}\left[q^{(3)}-sF\frac{\partial q^{(2)}}{\partial x}+V'(x)\frac{\partial q^{(1)}}{\partial x}+\frac{1}{2}s^{2}F^{2}\frac{\partial^{2}q^{(1)}}{\partial x^{2}}+V''(x)q^{(1)}-2sFV''(x)\frac{\partial\rho^{(0)}}{\partial x}-sFV'(x)\frac{\partial^{2}\rho^{(0)}}{\partial x^{2}}-\frac{1}{6}s^{3}F^{3}\frac{\partial^{3}\rho^{(0)}}{\partial x^{3}}-sFV'''(x)\rho^{(0)}\right]+O\left(\left(\frac{\tau}{\gamma}\right)^{2}\right)\Bigg\}.\end{aligned} \tag{28}$$

Equations (27) and (28) constitute perturbative expansions of the Perron-Frobenius equation. Comparing order by order in these expansions, the following relations ensue. For $O(\tau^{0})$:

$$\overline{\rho^{(0)}}=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\rho^{(0)}, \tag{29}$$

$O(\tau^{1/2})$:

$$\overline{q^{(1)}}=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\left[q^{(1)}-sF\frac{\partial\rho^{(0)}}{\partial x}\right], \tag{30}$$

$O(\tau)$:

$$\gamma^{-1}\overline{q^{(2)}}=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\left[\gamma^{-1}q^{(2)}-\gamma^{-1}sF\frac{\partial q^{(1)}}{\partial x}+\gamma^{-1}V'(x)\frac{\partial\rho^{(0)}}{\partial x}+\frac{1}{2}\gamma^{-1}s^{2}F^{2}\frac{\partial^{2}\rho^{(0)}}{\partial x^{2}}+\gamma^{-1}V''(x)\rho^{(0)}-\frac{\partial\rho^{(0)}}{\partial t}\right], \tag{31}$$

$O(\tau^{3/2})$:

$$\begin{aligned}\gamma^{-3/2}\overline{q^{(3)}}=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\Bigg[&\gamma^{-3/2}q^{(3)}-\gamma^{-3/2}sF\frac{\partial q^{(2)}}{\partial x}+\gamma^{-3/2}V'(x)\frac{\partial q^{(1)}}{\partial x}+\frac{1}{2}\gamma^{-3/2}s^{2}F^{2}\frac{\partial^{2}q^{(1)}}{\partial x^{2}}\\&+\gamma^{-3/2}V''(x)q^{(1)}-2\gamma^{-3/2}sFV''(x)\frac{\partial\rho^{(0)}}{\partial x}-\gamma^{-3/2}sFV'(x)\frac{\partial^{2}\rho^{(0)}}{\partial x^{2}}-\frac{1}{6}\gamma^{-3/2}s^{3}F^{3}\frac{\partial^{3}\rho^{(0)}}{\partial x^{3}}\\&-\gamma^{-3/2}sFV'''(x)\rho^{(0)}-\gamma^{-1/2}\frac{\partial q^{(1)}}{\partial t}+\gamma^{-1/2}sF\frac{\partial^{2}\rho^{(0)}}{\partial x\,\partial t}\Bigg],\end{aligned} \tag{32}$$

where the overbar in $\overline{\rho^{(0)}}$ and $\overline{q^{(i)}}$ serves to indicate its dependence on $\overline{F}$ instead of F. As we will show, these equations are important in the subsequent analysis as the lower order solution to the perturbative equation shall determine the next higher order solution, and leads to simplification in the form of Fokker-Planck equations, which are essential for the derivation of the directed current.

As the position space x is of special interest in the analysis on the macroscopic transport of the kicked particle, it is appropriate to integrate out the fluctuation F through marginal functions that depends on (x,t) in the following way:

$$P_{0}(x,t)\coloneqq\int dF\rho^{(0)}(F,x,t) \tag{33}$$

and

$$Q_{i}(x,t)\coloneqq\int dFq^{(i)}(F,x,t), \tag{34}$$

where $i=1,2,3$.

Next, let us further assume that G is a complete map [17] with phase space $[-1,1]$. This assumption shall simplify the set of equations given by Eqs. (29)–(32). To begin with, notice that these equations are of the following form:

$$\alpha(\overline{F},x,t)=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\beta(F,x,t), \tag{35}$$

with α denoting the functions on the left hand side of Eqs. (29)–(32) and β the functions on the right hand side.

A very useful property of complete maps is that they satisfy the integration lemma, which states that the marginal function of α and that of β are equal if Eq. (35) is satisfied. Mathematically, this is expressed as

$$\int_{-1}^{1}dF\alpha(F,x,t)=\int_{-1}^{1}dF\beta(F,x,t). \tag{36}$$

The proof of this theorem relates to an elementary property of the Perron-Frobenius operator [17].

Applying the integration lemma to Eq. (29) leads to $P_0(x,t)=P_0(x,t)$, which is trivial. However, when applied to Eq. (30), the result $\langle F\rangle=0$ is obtained. Thus, the ansatz given by Eq. (26) applies only to maps with vanishing means [17]. In other words, the chaotic force should have zero bias.

We next apply the integration lemma to Eq. (31) and obtain

$$\begin{aligned}\gamma^{-1}Q_2(x,t)=\gamma^{-1}Q_2(x,t)&-\gamma^{-1}s\frac{\partial}{\partial x}\int_{-1}^{1}dFFq^{(1)}\\&+\gamma^{-1}V'(x)\frac{\partial}{\partial x}\int_{-1}^{1}dF\rho^{(0)}\\&+\gamma^{-1}\frac{s^2}{2}\frac{\partial^2}{\partial x^2}\int_{-1}^{1}dFF^2\rho^{(0)}\\&+\gamma^{-1}V''(x)\int_{-1}^{1}dF\rho^{(0)}-\frac{\partial}{\partial t}\int_{-1}^{1}dF\rho^{(0)}.\end{aligned} \tag{37}$$

Based on the fact that Eq. (29) can be solved by any function of the form (see Appendix B)

$$\rho^{(0)}(F,x,t)=h(F)P_0(x,t), \tag{38}$$

Eq. (37) reduces to

$$\begin{aligned}\frac{s^2\langle F^2\rangle}{2\gamma}\frac{\partial^2}{\partial x^2}P_0(x,t)+\frac{\partial}{\partial x}\left(\frac{V'(x)}{\gamma}P_0(x,t)\right)-\frac{\partial}{\partial t}P_0(x,t)\\=\frac{s}{\gamma}\frac{\partial}{\partial x}\int_{-1}^{1}dFFq^{(1)},\end{aligned} \tag{39}$$

which is a kind of inhomogeneous Fokker-Planck equation for the zeroth-order probability distribution $P_0(x,t)$ due to fluctuations from complete maps. In fact, Eq. (39) is a Smoluchowski equation with a source term $(s/\gamma)\times(\partial/\partial x)\int_{-1}^{1}dFFq^{(1)}$.

Finally, we would like to remark that Eq. (38) has the appearance that F and x are statistically independent. This is rather surprising since x depends deterministically on F, and hence, $h(F)$ is perceivably the only independent distribution. The point to note is that this seeming independence between F and x occurs for $\rho^{(0)}$, which is the zeroth-order approximation to the true solution ρ. In general, this independence does not occur in the true solution ρ, as is apparent from the fact that the higher-order perturbative solutions $\rho^{(k)}=\rho^{(0)}+\Sigma_{i=1}^{k}(\tau/\gamma)^{i/2}q^{(i)}$ will not factorize with $h(F)$. For example, the $q^{(i)}$'s of Eqs. (30)–(32) are not separable into the form of $h(F)Q_i(x)$. However, in the next section, when the map is restricted to be double symmetric, we show that perturbative solution up to first-order, i.e., $\rho^{(1)}$, can be factorized with $h(F)$ owing to the symmetric properties of the double symmetric map.

IV. CHAOTIC FLUCTUATIONS FROM DOUBLE SYMMETRIC MAPS

In a subsequent analysis, we shall focus on a subclass of the complete maps, known as the double symmetric maps, which possess the following properties:

$$G(F)=G(-F), \tag{40}$$

$$h(F)=h(-F). \tag{41}$$

Examples of double symmetric maps are the even order Tchebyscheff maps as well as maps that are their conjugates [21]. Denoting $G^{(i)}$ as the ith-order Tchebyscheff map, the functional form of this map can be derived from the following iterative equation:

$$G^{(i+1)}(F)=2FG^{(i)}(F)-G^{(i-1)}(F). \tag{42}$$

With $G^{(0)}(F)=1$ and $G^{(1)}(F)=F$, it is simple to deduce that the second and fourth-order Tchebyscheff maps are $G^{(2)}(F)=2F^2-1$ and $G^{(4)}(F)=8F^4-8F^2+1$, respectively. Note that the Ulam map is the negative of the second-order Tchebyscheff map.

An important consequence of the properties of the double symmetric map is the following result:

$$\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}F^ih(F)=0, \tag{43}$$

with i being an odd integer, as it will lead to many simplifications. This relation holds because if F is a preimage, so is $-F$.

If we were to apply Eq. (43) with Eq. (38) to Eq. (30), we notice that the second term of Eq. (30) no longer contributes, and the equation simplifies to

$$\overline{q^{(1)}}=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}q^{(1)}. \tag{44}$$

In this case, $q^{(1)}$ can be expressed in the separable form (see Appendix B)

$$q^{(1)}(F,x,t)=h(F)Q_1(x,t). \tag{45}$$

Inserting Eq. (45) into Eq. (39) and noting that $\langle F\rangle=0$, the differential equation for the zeroth-order probability distribution reduces to, interestingly, the Smoluchowski equation

$$\frac{s^2\langle F^2\rangle}{2\gamma}\frac{\partial^2}{\partial x^2}P_0(x,t)+\frac{\partial}{\partial x}\left(\frac{V'(x)}{\gamma}P_0(x,t)\right)-\frac{\partial}{\partial t}P_0(x,t)=0. \tag{46}$$

Furthermore, for double symmetric maps, we can also employ Eq. (38), (43), and (45) to simplify Eq. (32), leading to

$$\gamma^{-1}\overline{q^{(3)}}=\sum_{F\in G^{-1}(\bar{F})}\frac{1}{|G'(F)|}\left[\gamma^{-1}q^{(3)}-\gamma^{-1}sF\frac{\partial q^{(2)}}{\partial x}+\gamma^{-1}V'(x)h(F)\frac{\partial Q_1}{\partial x}+\frac{1}{2}\gamma^{-1}s^2F^2h(F)\frac{\partial^2 Q_1}{\partial x^2}+\gamma^{-1}V''(x)h(F)Q_1-h(F)\frac{\partial Q_1}{\partial t}\right]. \tag{47}$$

By applying the integration lemma to Eq. (47), we obtain

$$\frac{s^2\langle F^2\rangle}{2\gamma}\frac{\partial^2}{\partial x^2}Q_1(x,t)+\frac{\partial}{\partial x}\left(\frac{V'(x)}{\gamma}Q_1(x,t)\right)-\frac{\partial}{\partial t}Q_1(x,t)=\frac{s}{\gamma}\frac{\partial}{\partial x}\int_{-1}^{1}dFFq^{(2)}. \tag{48}$$

With our interest in chaotic fluctuations based on double symmetric map in the limit $\tau/\gamma\to 0$, it is sufficient for us to consider position probability distribution up till the first order in τ/γ, i.e.,

$$P_1(x,t)=P_0(x,t)+\left(\frac{\tau}{\gamma}\right)^{1/2}Q_1(x,t), \tag{49}$$

which comes from [see Eq. (26)]

$$\rho^{(1)}(F,x,t)=\rho^{(0)}(F,x,t)+\left(\frac{\tau}{\gamma}\right)^{1/2}q^{(1)}(F,x,t)=h(F)P_1(x,t), \tag{50}$$

resulting from Eqs. (38) and (45). Then, by multiplying Eq. (48) with $(\tau/\gamma)^{1/2}$ and adding to Eq. (46), noting as well that $s^2\langle F^2\rangle=2kT$, we arrive at the inhomogeneous differential equation for the first-order position probability distribution of the particle

$$\frac{kT}{\gamma}\frac{\partial^2}{\partial x^2}P_1(x,t)+\frac{\partial}{\partial x}\left(\frac{V'(x)}{\gamma}P_1(x,t)\right)-\frac{\partial}{\partial t}P_1(x,t)=\tau^{1/2}\gamma^{-3/2}\frac{\partial}{\partial x}\int_{-1}^{1}dFsFq^{(2)}. \tag{51}$$

Remarkably, we again arrive at a Smoluchowski equation with a source term. More significantly, this source term is physically responsible for the flow of current in a spatially periodic potential, which will be shown in the next section. It is also important to note from Eq. (51) that in the limit $\tau/\gamma\to 0$, the position probability density function obeys the Smoluchowski equation, with $P_1(x)\to Z^{-1}\exp[-V(x)/kT]$, where Z is the normalization constant.

V. DIRECTED CURRENT FROM PERIODIC POTENTIAL

In this section, we are concerned with the derivation of a general analytical expression for the directed current, when the potential is periodic, and the nonequilibrium chaotic fluctuation is generated by double symmetric maps. To obtain a mathematical description of the directed current, which occurs in the steady state, we shall first set $\partial P_1/\partial t=0$ and integrate Eq. (51) to get

$$\frac{kT}{\gamma}\frac{\partial P_1}{\partial x}+\frac{V'(x)}{\gamma}P_1=\tau^{1/2}\gamma^{-3/2}\int_{-1}^{1}dFsFq^{(2)}-N_1, \tag{52}$$

where N_1 is a constant of integration. This first-order differential equation can be solved, and the solution is given by

$$P_1(x)=e^{-V(x)/kT}\left\{\left(\frac{\tau}{\gamma}\right)^{1/2}\frac{1}{kT}\int_{-1}^{x}e^{V(x')/kT}\times\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'-\frac{\gamma N_1}{kT}\int_{-1}^{x}e^{V(x')/kT}dx'+N_2\right\}, \tag{53}$$

in which the arbitrary constants of integration N_1 and N_2 are to be determined from the periodic boundary condition and the normalization condition, respectively.

Next, let us assume that the potential field has a period of 2, or

$$V(x+2n)=V(x). \tag{54}$$

Hence, x can be treated as an angle variable with the consequence that the density ρ must be periodic [22], i.e.,

$$\rho(F,x+2n)=\rho(F,x). \tag{55}$$

This implies, from Eq. (26), that

$$\rho^{(0)}(F,x+2n)=\rho^{(0)}(F,x) \tag{56}$$

and

$$q^{(i)}(F,x+2n)=q^{(i)}(F,x). \tag{57}$$

Note that the variable t has been suppressed as we are concerned with the steady state.

Then, applying Eq. (56) and Eq. (57) to Eq. (50), we obtain the periodic condition for $P_1(x)$:

$$P_1(x+2n)=P_1(x). \tag{58}$$

Through Eq. (53), $P_1(x+2n)$ can be expressed in the following form (see Appendix C):

$$P_1(x+2n)=e^{-V(x)/kT}\frac{n}{kT}\left\{\left(\frac{\tau}{\gamma}\right)^{1/2}\int_{-1}^{1}e^{V(x')/kT}\times\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'-\gamma N_1\int_{-1}^{1}e^{V(x')/kT}dx'\right\}+P_1(x). \tag{59}$$

As the first term on the right of Eq. (59) must be zero due to the periodicity of $P_1(x)$, i.e.,

$$\left(\frac{\tau}{\gamma}\right)^{1/2}\int_{-1}^{1}e^{V(x')/kT}\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'-\gamma N_1\int_{-1}^{1}e^{V(x')/kT}dx'=0, \tag{60}$$

we have

$$N_1=\left(\frac{\tau}{\gamma}\right)^{1/2}\frac{1}{\gamma}\frac{\int_{-1}^{1}e^{V(x')/kT}\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'}{\int_{-1}^{1}e^{V(x')/kT}dx'}. \tag{61}$$

Thus,

$$P_1(x)=e^{-V(x)/kT}\left\{\left(\frac{\tau}{\gamma}\right)^{1/2}\frac{1}{kT}\int_{-1}^{x}e^{V(x')/kT}\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'-\left(\frac{\tau}{\gamma}\right)^{1/2}\frac{1}{kT}\frac{\left[\int_{-1}^{1}e^{V(x')/kT}\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'\right]\left[\int_{-1}^{x}e^{V(x')/kT}dx'\right]}{\int_{-1}^{1}e^{V(x')/kT}dx'}+N_2\right\}, \tag{62}$$

with N_2 obtained by normalizing this expression, i.e., $\int_{-1}^{1}P_1(x)dx=1$.

Let us next proceed to determine the directed current J from Eq. (19). Defining the current as

$$J:=\lim_{n\to\infty}\langle\langle x_{n+1}-x_n\rangle\rangle, \tag{63}$$

where $\langle\langle\cdots\rangle\rangle$ means taking expectation with respect to $\tilde{\rho}(x,F)$ in the steady state and

$$\tilde{\rho}(F,x)=\rho^{(0)}(F,x)+\left(\frac{\tau}{\gamma}\right)^{1/2}q^{(1)}(F,x)+O\left(\frac{\tau}{\gamma}\right), \tag{64}$$

we have

$$\begin{aligned}J&=\lim_{n\to\infty}\left\langle\left\langle\left(\frac{\tau}{\gamma}\right)^{1/2}sF_n-\left(\frac{\tau}{\gamma}\right)V'(x_n)\right\rangle\right\rangle\\&=\left(\frac{\tau}{\gamma}\right)^{1/2}\int_{-1}^{1}\int_{-1}^{1}\left[sF-\left(\frac{\tau}{\gamma}\right)^{1/2}V'(x)\right]\times\left[\rho^{(0)}(F,x)+\left(\frac{\tau}{\gamma}\right)^{1/2}q^{(1)}(F,x)+O\left(\frac{\tau}{\gamma}\right)\right]dxdF\\&=\left(\frac{\tau}{\gamma}\right)^{1/2}\int_{-1}^{1}\int_{-1}^{1}\left[sF-\left(\frac{\tau}{\gamma}\right)^{1/2}V'(x)\right]\times\left[P_0(x)h(F)+\left(\frac{\tau}{\gamma}\right)^{1/2}Q_1(x)h(F)+O\left(\frac{\tau}{\gamma}\right)\right]dxdF.\end{aligned} \tag{65}$$

Noting that the last equation can be further simplified due to $\langle F\rangle=0$, and taking the leading order terms in τ/γ, we are

led to the following general expression for the directed current:

$$J=-\left(\frac{\tau}{\gamma}\right)\int_{-1}^{1}V'(x)P_1(x)dx, \tag{66}$$

which clearly shows that the current is connected to an ensemble average of the force field [5]. Substituting Eq. (62) in Eq. (66), we finally arrive at the central result of this paper,

$$J=-\left(\frac{\tau}{\gamma}\right)^{3/2}\frac{1}{kT}\int_{-1}^{1}V'(x)e^{-V(x)/kT}\times\left\{\int_{-1}^{x}e^{V(x')/kT}\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'-\left[\int_{-1}^{1}e^{V(x')/kT}\int_{-1}^{1}dFsFq^{(2)}(F,x')dx'\right]\times\frac{\int_{-1}^{x}e^{V(x')/kT}dx'}{\int_{-1}^{1}e^{V(x')/kT}dx'}\right\}dx. \tag{67}$$

The normalization constants N_2 in $P_1(x)$ does not contribute to the current because

$$-\left(\frac{\tau}{\gamma}\right)\int_{-1}^{1}V'(x)N_2e^{-V(x)/kT}dx=\left(\frac{\tau kTN_2}{\gamma}\right)[e^{-V(x)/kT}]_{-1}^{1}=0. \tag{68}$$

Equation (67) gives the first-order analytical expression for the directed current when the fluctuations come from the class of chaotic double symmetric maps. It clearly shows that the directed current is attributed to the source term $\int_{-1}^{1}dFsFq^{(2)}$ of the inhomogeneous Smoluchowski equation (51), and also indicates that the current scale as $(\tau/\gamma)^{3/2}$. However, we would like to emphasize that it is possible for the source term to vanish. In that case, the current J shall depend on a scaling relation in terms of τ/γ with an exponent that is greater than 3/2. The exact details of this dependency require the evaluation of higher-order perturbative equations, which turn out to be rather complicated.

VI. TILTING RATCHET WITH SYMMETRIC COSINE POTENTIAL AND ASYMMETRIC FLUCTUATIONS GENERATED BY THE ULAM MAP

The results obtained in the previous section are applicable to the general class of double symmetric maps. In order to better understand the effect of such nonequilibrium chaotic fluctuations on directed motion, it is necessary to solve the source term $\int_{-1}^{1}dFsFq^{(2)}$ for a typical case. In this section, we present such a solution for a special double symmetric map: the Ulam map.

To begin, let us reduce Eq. (31) to the following form by employing Eq. (38), Eq. (43) and Eq. (45):

$$\gamma^{-1}\overline{q^{(2)}}=\sum_{F\in G^{-1}(\bar{F})}\frac{1}{|G'(F)|}\left\{\gamma^{-1}q^{(2)}+\gamma^{-1}h(F)\frac{\partial}{\partial x}[V'(x)P_0]+\frac{1}{2}\gamma^{-1}s^2F^2h(F)\frac{\partial^2P_0}{\partial x^2}-h(F)\frac{\partial P_0}{\partial t}\right\}. \tag{69}$$

We multiply Eq. (46) with $h(F)$, and apply $\Sigma_{F\in G^{-1}(\bar{F})}[1/|G'(F)|]$ to both sides, yielding

$$\sum_{F\in G^{-1}(\bar{F})}\frac{1}{|G'(F)|}\left\{\gamma^{-1}h(F)\frac{\partial}{\partial x}[V'(x)P_0]+\frac{1}{2}\gamma^{-1}s^2\langle F^2\rangle h(F)\frac{\partial^2P_0}{\partial x^2}-h(F)\frac{\partial P_0}{\partial t}\right\}=0. \tag{70}$$

Subtracting Eq. (69) by Eq. (70), a more concise relation between $\overline{q^{(2)}}$ and $q^{(2)}$ is attained as follows:

$$\overline{q^{(2)}}=\sum_{F\in G^{-1}(\bar{F})}\frac{1}{|G'(F)|}\left\{q^{(2)}+\frac{1}{2}h(F)s^2[F^2-\langle F^2\rangle]\frac{\partial^2P_0}{\partial x^2}\right\}. \tag{71}$$

We then proceed by making the following ansatz for $q^{(2)}$:

$$q^{(2)}=h(F)Q_2(x,t)+sFh(F)R_2(x,t). \tag{72}$$

Applying this ansatz to Eq. (71) and after some simplification, we arrive at

$$s\bar{F}h(\bar{F})R_2=-\sum_{F\in G^{-1}(\bar{F})}\frac{1}{|G'(F)|}\times\left\{\frac{\sigma^2}{2}h(F)s^2\left[1-\frac{F^2}{\sigma^2}\right]\frac{\partial^2P_0}{\partial x^2}\right\}, \tag{73}$$

which can be further reduced for fluctuations based on the Ulam map (with $\sigma^2=1/2$) to

$$sG(F)h(\bar{F})R_2=-\frac{1}{4}s^2G(F)\frac{\partial^2P_0}{\partial x^2}\sum_{F\in G^{-1}(\bar{F})}\frac{1}{|G'(F)|}h(F). \tag{74}$$

Canceling terms in Eq. (74) while noting that $s=2\sqrt{kT}$ in the case of Ulam map, we get

$$R_2=-\frac{1}{2}(kT)^{1/2}\frac{\partial^2P_0}{\partial x^2}. \tag{75}$$

Knowing that

$$P_0(x) = Z^{-1} e^{-V(x)/kT} \tag{76}$$

is the solution of Eq. (46) in the steady state, with

$$Z = \int_{-1}^{1} e^{-V(x)/kT} dx \tag{77}$$

being the normalization constant, we deduce that

$$\frac{\partial^2 P_0}{\partial x^2} = -\frac{1}{ZkT} e^{-V(x)/kT} \left[V''(x) - \frac{V'(x)^2}{kT} \right]. \tag{78}$$

Substituting this result into Eq. (75), we have

$$R_2(x) = \frac{1}{2Z(kT)^{1/2}} e^{-V(x)/kT} \left[V''(x) - \frac{V'(x)^2}{kT} \right]. \tag{79}$$

With these results, we are able to evaluate the source term $\int_{-1}^{1} dFsFq^{(2)}(F,x)$ as follows:

$$\begin{aligned} \int_{-1}^{1} dFsFq^{(2)}(F,x) &= \int_{-1}^{1} dFsFh(F)Q_2(x) + \int_{-1}^{1} dFs^2F^2h(F)R_2(x) \\ &= s\langle F\rangle Q_2(x) + s^2\langle F^2\rangle R_2(x) \\ &= \frac{(kT)^{1/2}}{Z} e^{-V(x)/kT} \left[V''(x) - \frac{V'(x)^2}{kT} \right]. \end{aligned} \tag{80}$$

Moreover, this implies that

$$\int_{-1}^{x} e^{V(x')/kT} \int_{-1}^{1} dFsFq^{(2)}(F,x')dx' = \frac{(kT)^{1/2}}{Z} \int_{-1}^{x} \left[V''(x') - \frac{V'(x')^2}{kT} \right] dx' \tag{81}$$

and

$$\int_{-1}^{1} e^{V(x')/kT} \int_{-1}^{1} dFsFq^{(2)}(F,x')dx' = \frac{(kT)^{1/2}}{Z} \int_{-1}^{1} \left[V''(x') - \frac{V'(x')^2}{kT} \right] dx'. \tag{82}$$

Substituting Eqs. (81) and (82) into Eq. (62), the first-order position probability distribution is found to have the following form:

$$P_1(x) = e^{-V(x)/kT} \left\{ \left(\frac{\tau}{\gamma kT} \right)^{1/2} \frac{1}{Z} \int_{-1}^{x} \left[V''(x') - \frac{V'(x')^2}{kT} \right] dx' - \left(\frac{\tau}{\gamma kT} \right)^{1/2} \frac{1}{Z} \frac{\int_{-1}^{1} \left[V''(x') - \frac{V'(x')^2}{kT} \right] dx' \int_{-1}^{x} e^{V(x'')/kT} dx''}{\int_{-1}^{1} e^{V(x)/kT} dx} + N_2 \right\}. \tag{83}$$

As a result, according to Eq. (66), the directed current of the particle activated by chaotic fluctuations from the Ulam map is given by

$$\begin{aligned} J = -\left(\frac{\tau}{\gamma} \right)^{3/2} \frac{1}{Z} \int_{-1}^{1} V'(x) e^{-V(x)/kT} \Bigg\{ & \int_{-1}^{x} \left[\frac{V''(x')}{(kT)^{1/2}} - \frac{V'(x')^2}{(kT)^{3/2}} \right] dx' - \int_{-1}^{1} \left[\frac{V''(x')}{(kT)^{1/2}} - \frac{V'(x')^2}{(kT)^{3/2}} \right] dx' \frac{\int_{-1}^{x} e^{V(x')/kT} dx'}{\int_{-1}^{1} e^{V(x')/kT} dx'} \Bigg\} dx. \end{aligned} \tag{84}$$

In order to appreciate the symmetry breaking effect of the nonequilibrium chaotic fluctuation, the periodic potential has to be spatially symmetric and time independent, with a resulting spatial force field that averages out to zero. For the sake of simplicity, let us select

$$V(x) = \frac{\mu}{2}(1 - \cos \pi x). \tag{85}$$

Evaluating each terms of Eq. (83) in accordance with Eq. (85), the following more explicit analytical expression for the first-order position probability distribution is determined:

$$P_1(x)=\frac{e^{\mu\cos\pi x/2kT}}{\int_{-1}^{1}e^{\mu\cos\pi x/2kT}dx}\left\{\left(\frac{\tau}{\gamma kT}\right)^{1/2}\left[\frac{\mu\pi}{2}\sin\pi x-\frac{\mu^2\pi^2}{8kT}x+\frac{\mu^2\pi}{16kT}\sin 2\pi x-\frac{\mu^2\pi^2}{8kT}+\frac{\mu^2\pi^2}{4kT}\frac{\int_{-1}^{x}e^{-\mu\cos\pi x'/2kT}dx'}{\int_{-1}^{1}e^{-\mu\cos\pi x'/2kT}dx'}\right]+N_2'\right\}. \tag{86}$$

Keeping in mind that constant terms such as $\mu^2\pi^2/8kT$ and N_2' will not contribute to the current, the first-order analytical expression for the directed current is given by

$$J=-\left(\frac{\tau}{\gamma}\right)^{3/2}\frac{\mu^2\pi^2}{4(kT)^{1/2}\int_{-1}^{1}e^{\mu\cos\pi x/2kT}dx}\times\int_{-1}^{1}e^{\mu\cos\pi x/2kT}\sin\pi x\times\left\{\sin\pi x-\frac{\mu\pi}{4kT}x+\frac{\mu}{8kT}\sin 2\pi x+\frac{\mu\pi}{2kT}\frac{\int_{-1}^{x}e^{-\mu\cos\pi x'/2kT}dx'}{\int_{-1}^{1}e^{-\mu\cos\pi x'/2kT}dx'}\right\}dx. \tag{87}$$

VII. DISCUSSION

A comparison of our theoretical result, as given in Eq. (87), with numerical simulation based on the QKP map, is illustrated in Figs. 1 and 2. The figures show a close match between the analytical and simulation results, thus verifying our theoretical approach. Indeed, a closer scrutiny on these figures reveals that the correspondence becomes better as $\tau/\gamma\to 0$, which is to be expected. As discussed previously, when $\tau/\gamma\to 0$, $P_1(x)\to Z^{-1}\exp(-V(x)/kT)$, which implies that the current vanishes in the scaling limit. Hence, our analytical and numerical results indicate a convergence to the symmetric zero current state via scaling behavior of the leading order correction with respect to τ/γ in the following way:

$$J\sim\left(\frac{\tau}{\gamma}\right)^{3/2}. \tag{88}$$

Although this scaling law applies to fluctuations from Ulam map [see Eq. (87) and Fig. 3], it is conceivable that a subclass of double symmetric map may generate the same scaling behavior, leading to one universality class of nonequilibrium chaotic fluctuations. In fact, for chaotic fluctuation based on the fourth-order Tchebyscheff map [21], a scaling with $(\tau/\gamma)^{5/2}$ was observed through numerical simulations (see Fig. 3), indicating a possible origination of another universality class. This suggests that in the vicinity of the zero current fixed point, there may exist different routes to universality due to chaotic noise from double symmetric maps. The eventual elucidation of these universality classes is an interesting subject for further research.

In another context, it is also interesting to investigate the manner in which J depends on the noise intensity kT of the chaotic fluctuations, the outcome of which is shown in Fig. 4. Again, the agreement between analytical and numerical simulation is observed. The figure shows that the current is a convex function of kT, implying the existence of an optimum level for the chaotic drive. This result is physically intuitive, with the current vanishes as the noise strength reduces, and tailing off more slowly at increasing noise intensity, analogous to results obtained from noise that is deterministic and temporally asymmetric [9]. In both of these cases, directed current has been accounted for by the presence of odd higher-order correlations. But in this paper, we have gone one step further in examining the quantitative details of how macroscopic order, in the form of unidirectional particle flow, can be created out of microscopic chaotic fluctuation.

From a general perspective, the theoretical result given in Eq. (84) is not restricted to potentials that are spatially symmetric. By applying the same formalism to a well-known asymmetric and periodic ratchet potential [5]

$$V(x)=\frac{\mu}{2d}\left\{\sin[\pi(x-x_0)]+\frac{1}{4}\sin[2\pi(x-x_0)]+d\right\}, \tag{89}$$

where $x_0=0.3807$ and $d=1.1009$, the analytical result is again validated (see Fig. 4). More importantly, the asymmetry in the potential was found to enhance the directed current in this case.

In summary, this work provides exact analytical expressions up to leading order in τ/γ obtained from a perturbative Perron-Frobenius approach. We hope that these theoretical results will be useful for further research into noise-assisted energy transduction processes in the realm of science and engineering, where nonequilibrium chaotic fluctuations may be ubiquitous.

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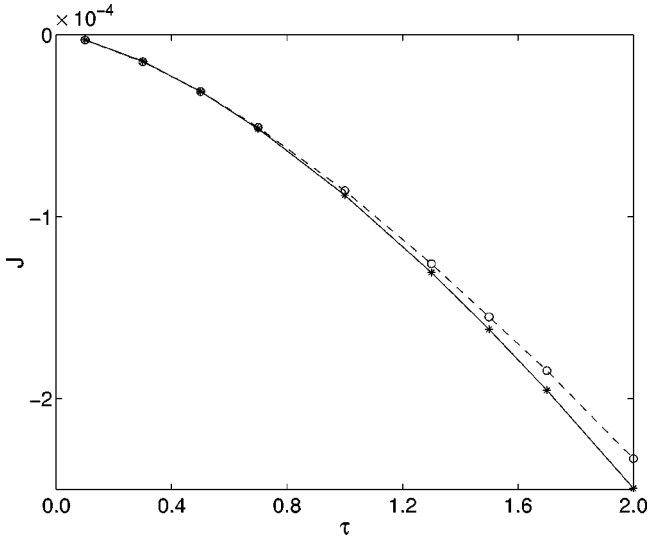

FIG. 1. The current versus τ for the symmetric cosine potential based on analytical expression (87) (solid curve with asterisks) and numerical simulation (dotted curve with open circles) for chaotic fluctuations from the Ulam map. The parameters in dimensionless units are $m=1.0$, $\gamma=200.0$, $\mu=1.0$, $kT=0.2$. The ensemble size used in the numerical simulation is 5×10^4, with an iteration length of 5×10^5.

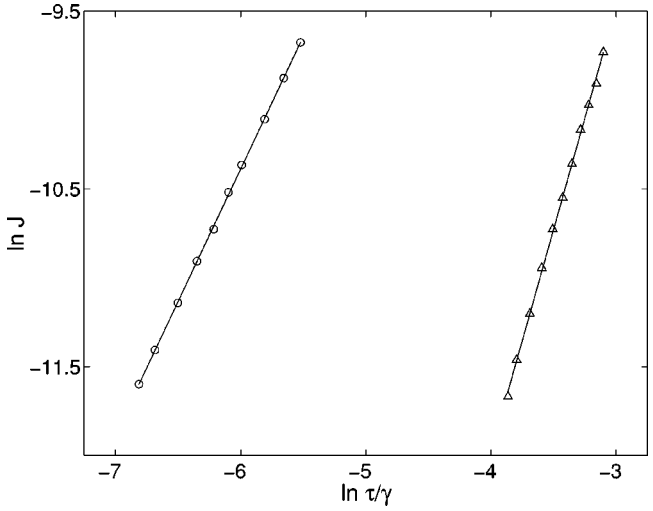

FIG. 3. The linear regression between $\ln J$ and $\ln \tau/\gamma$ from numerical simulation for the symmetric cosine potential when the chaotic fluctuation is based on the Ulam map (open circles) and the fourth-order Tchebyscheff map (open triangles markers). The gradient of the straight line is found to be 1.490 and 2.503, respectively. The parameters in dimensionless units are $m=1.0$, $\mu=1.0$, $kT=0.2$. The ensemble size used in the numerical simulation is 5×10^4, with an iteration length of 5×10^5.

APPENDIX A

In this appendix, we shall derive the discrete-time dynamical equations of p_n and x_n from Eqs. (2) and (5). For $n\tau^+ \leqslant t < (n+1)\tau$, Eq. (5) does not contain the impulsive force. In this case, the Hamilton's equations become

$$\frac{dx}{dt}=\frac{p}{m}, \tag{A1}$$

$$\frac{dp}{dt}=-\gamma p - V'(x), \tag{A2}$$

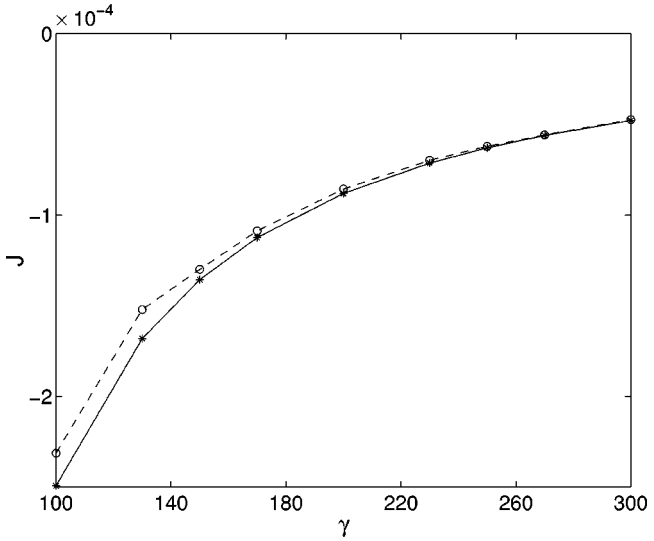

FIG. 2. The current versus γ for the symmetric cosine potential based on analytical expression Eq. (87) (solid curve with asterisks) and numerical simulation (dotted curve with open circles) for chaotic fluctuations from the Ulam map. The parameters in dimensionless units are $m=1.0$, $\tau=1.0$, $\mu=1.0$, $kT=0.2$. The ensemble size used in the numerical simulation is 5×10^4, with an iteration length of 5×10^5.

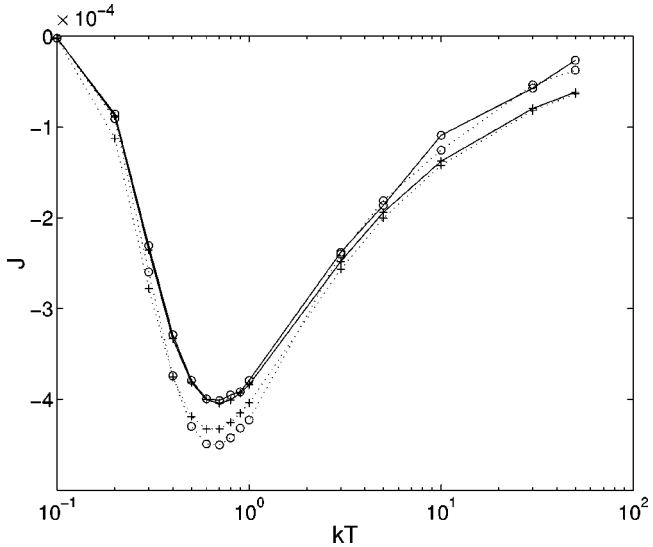

FIG. 4. The current versus kT for the symmetric cosine (solid curve) and asymmetric ratchet (dotted curve) potential based on analytical expression Eq. (84) (crosses) and numerical simulation (open circles) for chaotic fluctuations from the Ulam map. The parameters in dimensionless units are $m=1.0$, $\tau=1.0$, $\gamma=200.0$, $\mu=1.0$. The ensemble size used in the numerical simulation is 5×10^4, with an iteration length of 5×10^5.

where $V'(x)=\partial V(x)/\partial x$. The physical conditions at $t=n\tau^+$ are given by

$$x_n=x(n\tau^+)=x(n\tau^-) \tag{A3}$$

and

$$p_n=p(n\tau^+)=p(n\tau^-)+(\gamma m\tau)^{1/2}F^{\mathrm{I}}(n\tau)=p(n\tau^-)+(\gamma m\tau)^{1/2}F_n^{\mathrm{I}}. \tag{A4}$$

Equation (A3) states the continuity condition for the position of the particle, while Eq. (A4) relates the discontinuous change in momentum due to the δ kick.

We first integrate Eq. (A2) over the time interval $n\tau^+ \leqslant t<(n+1)\tau$:

$$\int_{n\tau^+}^{(n+1)\tau^-} d(e^{\gamma t}p)=-\int_{n\tau^+}^{(n+1)\tau^-} e^{\gamma t'}V'(x)dt'. \tag{A5}$$

The left hand side of the equation is equal to

$$e^{\gamma(n+1)\tau^-}p[(n+1)\tau^-]-e^{\gamma n\tau^+}p(n\tau^+). \tag{A6}$$

After a rearrangement of the terms, the equation becomes

$$\begin{aligned} e^{-\gamma 0^+}&p[(n+1)\tau^-] \\ &=e^{-\gamma\tau^-}p_n-e^{-\gamma(n+1)\tau}\int_{n\tau^+}^{(n+1)\tau^-}e^{\gamma t'}V'(x)dt' \\ &=e^{-\gamma\tau^-}p_n-\int_{n\tau^+}^{(n+1)\tau^-}e^{\gamma[t'-(n+1)\tau]}V'(x)dt' \\ &=e^{-\gamma\tau^-}p_n-e^{-\gamma\tau}\int_{0^+}^{\tau^-}e^{\gamma t''}V'[x_n(t'')]dt''. \end{aligned}$$

In the last equation, we have performed a change of variable, $t'-(n+1)\tau=t''-\tau$. Substituting the expression on the left hand side with Eq. (A4), we obtain

$$e^{-\gamma 0^+}p_{n+1}=e^{-\gamma\tau^-}p_n-e^{-\gamma\tau}\int_{0^+}^{\tau^-}e^{\gamma t}V'[x_n(t)]dt+e^{-\gamma 0^+}(\gamma m\tau)^{1/2}F_{n+1}^{\mathrm{I}}. \tag{A7}$$

Since $e^{\pm\gamma 0^+}=1\pm\gamma 0^+ +\cdots$, and taking into account that 0^+ is an infinitesimally small number, we have

$$p_{n+1}=e^{-\gamma\tau}p_n-e^{-\gamma\tau}\int_{0^+}^{\tau^-}e^{\gamma t}V'[x_n(t)]dt+(\gamma m\tau)^{1/2}F_{n+1}^{\mathrm{I}}. \tag{A8}$$

Similarly, we integrate Eq. (A1) over the same time interval $n\tau^+\leqslant t<(n+1)\tau$ and perform the change of variable $t'-(n+1)\tau=t''-\tau$. The result is

$$x[(n+1)\tau^-]=x_n+\frac{1}{m}\int_{0^+}^{\tau^-}p_n(t'')dt''. \tag{A9}$$

In this equation, we have used the notation $p_n(t'')=p(t''+n\tau)$. With the continuity condition given by Eq. (A3), we obtain

$$x_{n+1}=x_n+\frac{1}{m}\int_{0^+}^{\tau^-}p_n(t)dt. \tag{A10}$$

APPENDIX B

In this appendix, we shall show that the solution of equation

$$\alpha(\overline{F},x,t)=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}\alpha(F,x,t), \tag{B1}$$

with F derived from chaotic maps which are ergodic, takes the separable form [17]

$$\alpha(F,x,t)=h(F)A(x,t), \tag{B2}$$

where

$$A(x,t)=\int dF\alpha(F,x,t). \tag{B3}$$

To begin, it is essential to note that Eq. (B1) relates between two identical functions with respect to the variables $\overline{F}$ and F, while the values of x and t are the same at both sides of the equation. As the chaotic maps in consideration are ergodic, an invariant density $h(F)$ exists and is *unique* [18]. It also satisfies

$$h(\overline{F})=\sum_{F\in G^{-1}(\overline{F})}\frac{1}{|G'(F)|}h(F). \tag{B4}$$

This implies that for Eq. (B1) to be true for all x and t, $\alpha(F,x,t)$ must have the separable form given by Eq. (B2). Otherwise, different values of x and t will lead to different functional forms of $h(F)$, which contradict the uniqueness of the invariant density.

APPENDIX C

In this appendix, we shall derive Eq. (59). This requires us to evaluate a few quantities. For the first quantity, we have

$$\int_{-1+2(\eta-1)}^{1+2(\eta-1)}e^{V(x')/kT}dx'=\int_{-1}^{1}e^{V(x)/kT}dx, \tag{C1}$$

as a result of Eq. (54) and a change of variable of the form $x=x'-2(\eta-1)$, where η is an integer. This result is useful for the following calculation:

$$\int_{-1}^{x+2n} e^{V(x')/kT}dx' = \sum_{\eta=1}^{n} \int_{-1+2(\eta-1)}^{1+2(\eta-1)} e^{V(x')/kT}dx' + \int_{-1+2n}^{x+2n} e^{V(x')/kT}dx' = n\int_{-1}^{1} e^{V(x'')/kT}dx'' + \int_{-1}^{x} e^{V(x'')/kT}dx'', \tag{C2}$$

where a change of variable $x''=x'-2n$ has been performed on the last integral. In addition, Eq. (57) (with $i=2$) implies that

$$\int_{-1+2(\eta-1)}^{1+2(\eta-1)} e^{V(x')/kT}\int_{-1}^{1} dFsFq^{(2)}(F,x')dx' = \int_{-1}^{1} e^{V(x)/kT}\int_{-1}^{1} dFsFq^{(2)}(F,x)dx, \tag{C3}$$

when $x=x'-2(\eta-1)$. Using Eq. (C3) and applying similar techniques to the determination of Eq. (C2) to the following integral, we obtain

$$\int_{-1}^{x+2n} e^{V(x')/kT}\int_{-1}^{1} dFsFq^{(2)}(F,x')dx' = n\int_{-1}^{1} e^{V(x'')/kT}\int_{-1}^{1} dFsFq^{(2)}(F,x'')dx'' + \int_{-1}^{x} e^{V(x'')/kT}\int_{-1}^{1} dFsFq^{(2)}(F,x'')dx''. \tag{C4}$$

Employing the two evaluated quantities (C2) and (C4) on $P_1(x+2n)$ from Eq. (53), Eq. (59) is obtained.

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